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Fully developed laminar forced convection in circular ducts for power-law fluids with viscous dissipation

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Abstract—The asymptotic behaviour of the temperature field for the laminar and hydrodynamically developed forced convection of a power-law fluid which flows in a circular duct is studied. The effect of viscous dissipation is taken into account, while the axial heat conduction in the fluid is considered as negligible. First, it is shown that the sufficient condition for the existence of an asymptotically invariant value of the Nusselt number is less restrictive than that usually employed in the previous literature. Then, both the asymptotic Nusselt number and the asymptotic behaviour of the temperature field for the axial wall heat flux distributions which yield a thermally developed region are determined. Finally, the asymptotic Nusselt number and the asymptotic temperature distribution are evaluated analytically in the cases of either uniform wall temperature or convection with an external isothermal fluid. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

The laminar forced convection of power-law fluids in circular ducts has been investigated by many authors and the most interesting results obtained in this field are reviewed in refs. [1–3].

Many authors deal with laminar forced convection models which neglect both the effect of viscous dissipation and that of axial heat conduction in the fluid. For instance, Grigull [4] shows that the fully developed value of the Nusselt number for a power-law fluid flowing in a circular tube with a uniform heat flux is given by

$$Nu_{H\infty} = \frac{8(3n+1)(5n+1)}{31n^2+12n+1}, \qquad (1)$$

where n is the power-law index. For a Newtonian fluid, i.e. for $n = 1$, equation (1) yields the well known result $Nu_{H\infty} = 48/11$. Also, the thermal entrance region is studied by various authors. The temperature field and the local Nusselt number in the thermal entrance region for uniform wall temperature is evaluated by Cotta and Özisik [5] and by Prusa and Manglik [6] in the case of negligible viscous dissipation and axial heat conduction in the fluid. Under the same assumptions, the thermal entrance region in the case of uniform wall heat flux is studied in ref. [7].

The effect of viscous dissipation in the thermal entrance region is investigated by Liou and Wang [8], Berardi *et al.* [9], Lawal and Mujumdar [10], and Dang [11] who also takes into account the effect of the axial heat conduction in the fluid. Liou and Wang [8] consider a boundary condition of uniform wall heat flux, Berardi *et al.* [9] assume wall heat transfer by convection with an external isothermal fluid, while Lawal and Mujumdar [10] and Dang [11] prescribe a uniform wall temperature.

In the analysis of laminar forced convection of Newtonian fluids flowing in circular ducts, it has been shown that the effect of viscous dissipation is very relevant in the fully developed region, both if the wall temperature is uniform and if convection with an external isothermal fluid occurs [12, 13]. In fact, if the wall temperature is uniform, the fully developed value of the Nusselt number is $48/5 = 9.6$ if viscous dissipation is taken into account [12], while it is equal to 3.6568 if viscous dissipation is neglected [14]. In ref. [12], it is shown that the value 48/5 holds for any value of the parameters which characterize the system, in particular for very small values of the dynamic viscosity of the fluid. This result leads to the following conclusion: when the wall temperature is uniform the effect of viscous dissipation cannot be neglected in the fully developed region. The same conclusion can be taken if convective boundary conditions are prescribed at the tube wall. In fact, with these boundary conditions and if viscous dissipation is taken into account, the fully developed value of the Nusselt number is 48/5 for every value of the Biot number and of the other parameters [13]. On the other hand, it is well known that, if a forced convection model with no viscous dissipation is employed, the fully developed value of the Nusselt number for convective boundary conditions depends on the value of the Biot number [15]. As pointed out by Lawal and Mujumdar [10], in the case of uniform wall temperature, viscous dissipation strongly modifies not only the fully developed

NOMENCLATURE

A	$= T_{01b} - T_{02b}$, constant
$a(R)$	solution of equations (47) and (48)
Bi	$= h_e r_0/k$, Biot number
$Br(X)$	local Brinkman number, defined by equation (22)
$Br_\infty^{(s)}$	singular value of Br_∞ evaluated by equation (41)
C	dimensionless constant employed in equation (46)
f	function of R defined either by equation (27) or by equation (35)
$\mathscr{F}$	function defined in equation (13)
h_e	convection coefficient with a fluid external to the tube wall
k	thermal conductivity
n	power-law index
Nu	Nusselt number, $2r_0 q_w/[k(T_w - T_b)]$
Pe	Peclet number, $2\bar{u}r_0/\alpha$
q_w	wall heat flux
r	radial coordinate
r_0	radius of the tube
R	$= r/r_0$, dimensionless radial coordinate
R'	dummy integration variable
T	temperature
T_1, T_2	temperature fields
T_0, T_{01}, T_{02}	inlet temperature distributions
T_f	reference temperature of a fluid external to the tube wall
$\tilde{T}$	$= T_1 - T_2$, temperature field
u	velocity component in the axial direction
$\bar{u}$	mean value of u
x	axial coordinate
X	$= x/(2r_0 Pe)$, dimensionless axial coordinate
X'	dummy integration variable
Y	arbitrary function of r and x.

Greek symbols

α	thermal diffusivity
β	dimensionless parameter defined in equation (43)
η	consistency factor employed in equation (2)
ϑ	dimensionless temperature defined in equation (20)
ϑ_f	$= k r_0^{n-1}(T_f - T_{0b})/(\eta \bar{u}^{n+1})$, dimensionless parameter
Θ	$= (T_w - T)/(T_w - T_b)$, dimensionless function of R and X
τ_{rx}	rx component of the stress tensor.

Subscripts

b	bulk quantity
H	quantity evaluated for uniform wall heat flux
T	quantity evaluated for uniform wall temperature
w	quantity evaluated at the wall
∞	quantity evaluated for $X \to +\infty$.

value of the Nusselt number, but the temperature distribution in the whole thermal entrance region.

The aim of this paper is to study how viscous dissipation affects the fully developed behaviour of laminar forced convection for a power-law fluid which flows in a circular tube with a prescribed axial distribution of wall heat flux. In order to have an asymptotically invariant value of the Nusselt number in the fully developed region, some restrictions must be imposed on the asymptotic behaviour of the axial distribution of wall heat flux. However, in this paper it is shown that these restrictions are not so strong as it is usually retained in the literature. In fact, in the literature it is commonly accepted, at least for Newtonian fluids, that an axial distribution of wall heat flux yields a fully developed region if it behaves asymptotically as an exponential function of the axial coordinate [14–17]. In this paper, many axial distributions of wall heat flux are considered which cannot be approximated asymptotically by an exponential function and which, nonetheless, provide a fully developed value of the Nusselt number.

GOVERNING EQUATIONS

In this section, the energy balance equation and its boundary conditions are considered. It is shown that the boundary conditions at the inlet section affects the asymptotic behaviour of the temperature field only through an additive constant. Finally, the boundary value problem is written in a dimensionless form.

Let us consider a power-law fluid which steadily flows in a circular tube with radius r_0. The flow is supposed to be laminar and hydrodynamically developed. Moreover, the properties of the fluid are supposed to be constant. The stress–strain relationship for power-law fluids can be written as

$$\tau_{rx} = \eta \left|\frac{du}{dr}\right|^{n-1} \frac{du}{dr} \quad (2)$$

where the case $n = 1$ corresponds to a Newtonian fluid. Moreover, the axial component of the velocity field is given by [5]

$$u(r) = \frac{3n+1}{n+1}\bar{u}\left[1 - \left(\frac{r}{r_0}\right)^{(n+1)/n}\right]. \quad (3)$$

If the axial heat conduction in the fluid can be neglected, the energy balance equation can be expressed as

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{u(r)r}{\alpha}\frac{\partial T}{\partial x} - \frac{r}{k}\tau_{rx}\frac{du}{dr}. \quad (4)$$

The second term at the right hand side of equation (4) is due to the power generated by viscous dissipation. The boundary conditions for the temperature field are the following:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0, \quad k\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = q_w(x) \quad (5)$$

$$T(r,0) = T_0(r). \quad (6)$$

The forced convection problem leads to a fully developed region for large values of x if [18]

$$\operatorname*{Lim}_{x\to+\infty} \frac{T_w(x) - T(r,x)}{T_w(x) - T_b(x)} = \operatorname*{Lim}_{x\to+\infty} \Theta(r/r_0, x/(2r_0 Pe)) = \Theta_\infty(r/r_0), \quad (7)$$

where $\Theta_\infty(r/r_0)$ is a continuous and differentiable function of r, the Peclet number is given by $Pe = 2\bar{u}r_0/\alpha$ and the bulk value of an arbitrary function $Y(r,x)$ is defined as

$$Y_b(x) = \frac{2}{\bar{u}r_0^2}\int_0^{r_0} Y(r,x)u(r)r\,dr. \quad (8)$$

It is well known [18] that, if condition (7) holds, then a fully developed value of the Nusselt number exists and is given by

$$\operatorname*{Lim}_{x\to+\infty} Nu = 2r_0 \operatorname*{Lim}_{x\to+\infty} \frac{\left.\frac{\partial T}{\partial r}\right|_{r=r_0}}{T_w(x) - T_b(x)} = -2r_0\left.\frac{d\Theta_\infty}{dr}\right|_{r=r_0} = Nu_\infty. \quad (9)$$

The boundary value problem, expressed by equations (4)–(6) has a unique solution. In fact, it can be easily checked that, if $T_1(r,x)$ and $T_2(r,x)$ are two temperature fields which solve the boundary value problem (4)–(6), the function $\tilde{T}(r,x) = T_1(r,x) - T_2(r,x)$ is a solution of the boundary value problem

$$\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{T}}{\partial r}\right) = \frac{u(r)r}{\alpha}\frac{\partial \tilde{T}}{\partial x}, \quad (10)$$

$$\left.\frac{\partial \tilde{T}}{\partial r}\right|_{r=0_0} = 0, \quad \left.\frac{\partial \tilde{T}}{\partial r}\right|_{r=r_0} = 0 \quad (11)$$

$$\tilde{T}(r,0) = 0. \quad (12)$$

Let us define a function $\mathscr{F}(x)$ as follows:

$$\mathscr{F}(x) = \int_0^{r_0} \tilde{T}(r,x)^2 u(r)r\,dr. \quad (13)$$

Obviously, on account of equation (13), $\mathscr{F}(x) \geqslant 0$ for every $x \geqslant 0$. In particular, equations (12) and (13) ensure that $\mathscr{F}(0) = 0$. The derivative of $\mathscr{F}(x)$ is given by

$$\frac{d\mathscr{F}(x)}{dx} = 2\int_0^{r_0} \tilde{T}(r,x)\frac{\partial \tilde{T}(r,x)}{\partial x}u(r)r\,dr. \quad (14)$$

By employing equation (10), equation (14) can be rewritten as

$$\frac{d\mathscr{F}(x)}{dx} = 2\alpha\int_0^{r_0} \tilde{T}(r,x)\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{T}(r,x)}{\partial r}\right)dr. \quad (15)$$

An integration by parts of the right hand side of equation (15) yields

$$\frac{d\mathscr{F}(x)}{dx} = 2\alpha\left[r\tilde{T}(r,x)\frac{\partial \tilde{T}(r,x)}{\partial r}\right]_0^{r_0} - 2\alpha\int_0^{r_0} r\left(\frac{\partial \tilde{T}(r,x)}{\partial r}\right)^2 dr. \quad (16)$$

On account of equations (11) and (16), one obtains

$$\frac{d\mathscr{F}(x)}{dx} = -2\alpha\int_0^{r_0} r\left(\frac{\partial \tilde{T}(r,x)}{\partial r}\right)^2 dr. \quad (17)$$

Obviously, equation (17) ensures that $d\mathscr{F}(x)/dx \leqslant 0$ for every $x \geqslant 0$ i.e. that $\mathscr{F}(x)$ is a monotonically decreasing function of x. Since $\mathscr{F}(0) = 0$, one can conclude that $\mathscr{F}(x) \leqslant 0$ for every $x \geqslant 0$. On the other hand, as a consequence of equation (13), $\mathscr{F}(x)$ is nonnegative. Therefore, $\mathscr{F}(x) = 0$ for every $x \geqslant 0$. The integral on the right hand side of equation (13) vanishes only if the integrand is zero, i.e. if $\tilde{T}(r,x) = 0$ for every r and for every x.

This proof is similar to that of the uniqueness theorem for the initial value problem of heat conduction presented in Carslaw and Jaeger [19]. It can be easily proved that the uniqueness of the solution of the boundary value problem (4)–(6) holds also if the boundary condition at $r = r_0$ is of the first kind, i.e. if the prescribed quantity at $r = r_0$ is the temperature.

Let $q_w(x)$ be prescribed. Then, for every inlet temperature distribution $T_0(r)$, equations (4)–(6) yield a unique temperature field within the duct. However, two different inlet temperature distributions $T_{01}(r)$ and $T_{02}(r)$ yield two temperature fields within the duct, $T_1(r,x)$ and $T_2(r,x)$, such that asymptotically, i.e. for large values of x, their difference is a constant A. In fact, it can be easily checked that function $\tilde{T}(r,x)$

$= T_1(r,x) - T_2(r,x)$ fulfils equation (10), (11) and the boundary condition at $x = 0$

$$\tilde{T}(r,0) = T_{01}(r) - T_{02}(r). \quad (18)$$

Equations (10), (11) and (18) show that $\tilde{T}(x,r)$ can be interpreted as the temperature field of a fluid with no viscous dissipation flowing within a tube with an adiabatic wall and a nonuniform inlet temperature profile $T_{01}(r) - T_{02}(r)$. It is well known that, for any radial distribution of the inlet temperature, if the wall is adiabatic and no heat generation occurs within the fluid, the temperature field tends to become uniform at sections sufficiently distant from $x = 0$. Therefore, in order to determine the asymptotic behaviour of the temperature field within the duct, it is sufficient to find one of the infinite solutions of equations (4) and (5); then, one can account for the inlet temperature distribution by adding to this solution a suitable constant A. However, since both the function $\Theta(r,x)$ and the local Nusselt number Nu are defined through temperature differences, their asymptotic expressions are independent of the constant A, and then of the inlet temperature distribution. By recalling that $\tilde{T}(x,r)$ is the temperature field of a fluid with no viscous dissipation flowing within a tube with an adiabatic wall, it is easily verified that its bulk value $\tilde{T}_b(x)$ is a constant, $\tilde{T}_b(x) = \tilde{T}_b(0) = T_{01b} - T_{02b}$. Therefore, since $\tilde{T}(x,r)$ tends to a constant value A, one can conclude that $A = \tilde{T}_b(0) = T_{01b} - T_{02b}$.

On account of equations (2) and (3), equation (4) can be rewritten as

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{3n+1}{n+1}\frac{\bar{u}r}{\alpha}\left[1 - \left(\frac{r}{r_0}\right)^{(n+1)/n}\right]\frac{\partial T}{\partial x} - \frac{\eta\bar{u}^{n+1}}{kr_0^n}\left(\frac{3n+1}{n}\right)^{n-1}\left(\frac{r}{r_0}\right)^{(2n+1)/n}. \quad (19)$$

If T_{0b} is the bulk value of the inlet temperature profile, by introducing the dimensionless coordinates $R = r/r_0$, $X = x/(2r_0 Pe)$ and the dimensionless temperature

$$\vartheta = kr_0^{n-1}\frac{T - T_{0b}}{\eta\bar{u}^{n+1}}, \quad (20)$$

equation (19) can be written in the dimensionless form

$$\frac{\partial}{\partial R}\left(R\frac{\partial\vartheta}{\partial R}\right) = \frac{3n+1}{4(n+1)}R(1 - R^{(n+1)/n})\frac{\partial\vartheta}{\partial X} - \left(\frac{3n+1}{n}\right)^{n+1}R^{(2n+1)/n}. \quad (21)$$

As a consequence of the analysis performed above, it is easily verified that the dimensionless temperature ϑ defined by equation (20) becomes independent of the inlet temperature profile for large values of x. Since we are dealing with an arbitrary axial wall heat flux distribution, the Brinkman number is a local parameter defined as

$$Br(X) = \frac{\eta\bar{u}^{n-1}}{(2r_0)^n q_w(x)}. \quad (22)$$

On account of equations (20) and (22), equation (5) can be rewritten as

$$\left.\frac{\partial\vartheta}{\partial R}\right|_{R=0} = 0, \quad \left.\frac{\partial\vartheta}{\partial R}\right|_{R=1} = \frac{1}{2^n Br(X)}. \quad (23)$$

By integrating both sides of equation (21) with respect to R in the interval $0 \leqslant R \leqslant 1$ and by employing equation (23), it is easily proved that the bulk value of the dimensionless temperature ϑ fulfils the equation

$$\frac{d\vartheta_b(X)}{dX} = \frac{2^{3-n}}{Br(X)} + 8\left(\frac{3n+1}{n}\right)^n. \quad (24)$$

Since equation (20) ensures that $\vartheta_b(0) = 0$, equation (24) can be easily integrated and yields

$$\vartheta_b(X) = 2^{3-n}\int_0^X \frac{dX'}{Br(X')} + 8\left(\frac{3n+1}{n}\right)^n X. \quad (25)$$

ASYMPTOTIC BEHAVIOUR OF THE TEMPERATURE FIELD

In this section, three classes of functions $Br(X)$ which yield an asymptotic fully developed region are considered. In these three cases, the asymptotic behaviour of the dimensionless temperature distribution ϑ is determined.

Let us first consider the axial distributions of wall heat flux such that

$$\lim_{X\to+\infty} Br(X) = \pm\infty. \quad (26)$$

Equation (26) is fulfilled by all the axial distributions of wall heat flux which tend to zero for $x \to +\infty$. In this case, a solution of equations (21) and (23) exists which, for large values of X, can be expressed as

$$\vartheta(R,X) = \vartheta_b(X) + f(R), \quad (27)$$

where $f(R)$ is a continuous and differentiable function of R. In fact, by substituting equation (27) in equations (21) and (23) and by employing equations (24) and (26), one obtains in the limit $X \to +\infty$

$$\frac{d}{dR}\left(R\frac{df}{dR}\right) = \left(\frac{3n+1}{n}\right)^{n+1}\frac{2nR - (3n+1)R^{(2n+1)/n}}{n+1} \quad (28)$$

$$\left.\frac{df}{dR}\right|_{R=0} = 0, \quad \left.\frac{df}{dR}\right|_{R=1} = 0. \quad (29)$$

Equation (27) implies that the bulk value of function $f(R)$ must be zero. Equations (28) and (29) and this additional constraint determine a unique function $f(R)$ given by

$$f(R) = \left(\frac{3n+1}{n}\right)^{n+1}\left[\frac{n}{2(n+1)}\left(R^2 - \frac{2n}{3n+1}R^{(3n+1)/n}\right) - \frac{n}{4(3n+1)}\right]. \quad (30)$$

On account of equations (20), (27) and (30), equation (7) is fulfilled and $\Theta_\infty(R)$ can be expressed as

$$\Theta_\infty(R) = \frac{f(1)-f(R)}{f(1)} \quad (31)$$

$$= 2 - \frac{2(3n+1)}{n+1}\left(R^2 - \frac{2n}{3n+1}R^{(3n+1)/n}\right).$$

Equations (9) and (31) yield the asymptotic value of Nu, i.e.

$$Nu_\infty = -2\left.\frac{d\Theta_\infty}{dR}\right|_{R=1} = 0. \quad (32)$$

Plots of function $\Theta_\infty(R)$ for various values of n are reported in Fig. 1. Equations (31) and (32) describe the asymptotic behaviour of the temperature field for any axial distribution of wall heat flux which tends to zero for $x \to +\infty$, as, for instance, the exponentially decreasing wall heat fluxes considered in refs. [14–17].

Let us now assume that the local Brinkman number fulfils the condition

$$\lim_{X\to+\infty} Br(X) = Br_\infty, \quad (33)$$

where Br_∞ is any real number, and, if $Br_\infty = 0$, the further condition

$$\lim_{X\to+\infty} \frac{1}{Br(X)}\frac{dBr(X)}{dX} = 0. \quad (34)$$

Equations (33) and (34) are satisfied by a uniform wall heat flux, by a linearly varying wall heat flux, and by many other axial distributions of wall heat flux expressed, for instance, by polynomial functions or by quotients of polynomials with the degree of the numerator not lower than the degree of the denominator. If equations (33) and (34) hold, there exists an asymptotic fully developed region for the temperature field. In fact, there exists a solution of equations (21) and (23) which for large values of X can be expressed as

$$\vartheta(R, X) = \vartheta_b(X) + \frac{f(R)}{Br(X)}, \quad (35)$$

where $f(R)$ is a continuous and differentiable function of R. In fact, by substituting equation (35) in equations (21) and (23) and by employing equations (24), (33) and (34), one obtains in the limit $X \to +\infty$

$$\frac{d}{dR}\left(R\frac{df}{dR}\right) = \frac{3n+1}{4(n+1)}\left\{\left[2^{3-n} + 8\left(\frac{3n+1}{n}\right)^n Br_\infty\right]R - \left[2^{3-n} + 4\left(\frac{3n+1}{n}\right)^{n+1} Br_\infty\right]R^{(2n+1)/n}\right\} \quad (36)$$

$$\left.\frac{df}{dR}\right|_{R=0} = 0, \quad \left.\frac{df}{dR}\right|_{R=1} = \frac{1}{2^n}. \quad (37)$$

Equations (36) and (37) determine a unique function $f(R)$ with a vanishing bulk value, i.e.

$$f(R) = -\frac{19n^2+8n+1}{2^{2+n}(5n+1)(3n+1)} - \frac{Br_\infty}{4}\left(\frac{3n+1}{n}\right)^n + \frac{3n+1}{4(n+1)}\left\{\frac{1}{4}\left[2^{3-n} + 8\left(\frac{3n+1}{n}\right)^n Br_\infty\right]R^2 - \left(\frac{n}{3n+1}\right)^2\left[2^{3-n} + 4\left(\frac{3n+1}{n}\right)^{n+1} Br_\infty\right]R^{(3n+1)/n}\right\}. \quad (38)$$

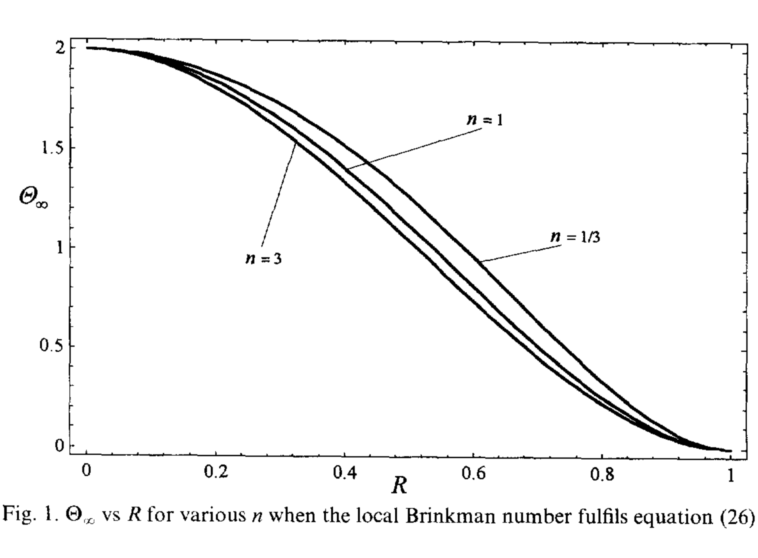

Fig. 1. Θ_∞ vs R for various n when the local Brinkman number fulfils equation (26).

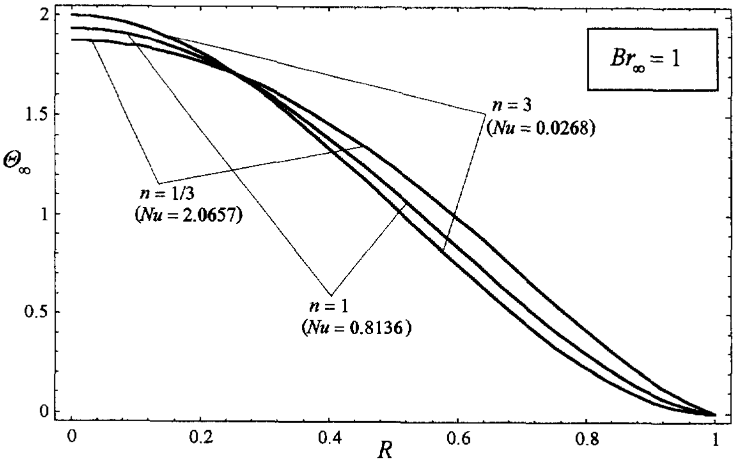

Fig. 2. Θ_∞ vs R for various n and for $Br_\infty = 1$ when the local Brinkman number fulfils equation (33).

On account of equations (20), (35) and (38), equation (7) is fulfilled and $\Theta_\infty(R)$ can be written as

$$\Theta_\infty(R) = \frac{f(1)-f(R)}{f(1)}$$
$$= \frac{3n+1}{4(n+1)}\left\{\frac{1}{4}\left[2^{3-n}+8\left(\frac{3n+1}{n}\right)^n Br_\infty\right]\right.$$
$$\times(1-R^2)-\left(\frac{n}{3n+1}\right)^2\left[2^{3-n}+4\left(\frac{3n+1}{n}\right)^{n+1} Br_\infty\right]$$
$$\left.\times\left(1-R^{(3n+1)/n}\right)\right\}\left[\frac{31n^2+12n+1}{2^{2+n}(3n+1)(5n+1)}\right.$$
$$\left.+\frac{Br_\infty}{4}\left(\frac{3n+1}{n}\right)^n\right]^{-1}. \quad (39)$$

By employing equations (9) and (39), the asymptotic value of Nu can be evaluated and can be expressed as

$$Nu_\infty = \left[\frac{31n^2+12n+1}{8(3n+1)(5n+1)}+2^{n-3}Br_\infty\left(\frac{3n+1}{n}\right)^n\right]^{-1}. \quad (40)$$

Equation (40) agrees with the results obtained by Liou and Wang [8] in the case of uniform wall heat flux. However, the range of validity of equation (40) is much broader and includes also every nonuniform wall heat flux distribution which fulfils equations (33) and (34). If the wall heat flux distribution is such that $Br_\infty = 0$, equation (40) coincides with equation (1) which has been obtained in the case of uniform wall heat flux and negligible viscous dissipation.

Plots of $\Theta_\infty(R)$ for various values of n and for Br_∞ equal to 1, 0 and -1 are reported in Figs. 2–4. In these figures, the asymptotic values of Nu, evaluated in each case by equation (40), are given. Figure 4 refers to a value of Br_∞ such that the asymptotic values of Nu are negative. Indeed, equations (39) and (40) reveal that, for any value of n, there exists a negative value of Br_∞ which produces a singularity in both the asymptotic value of Nu and in the expression of $\Theta_\infty(R)$. This singular value of Br_∞ is given by

$$Br_\infty^{(s)} = -2^{-n}\left(\frac{n}{3n+1}\right)^n\frac{31n^2+12n+1}{(3n+1)(5n+1)}. \quad (41)$$

A plot of $Br_\infty^{(s)}$ as a function of n is presented in Fig. 5. This plot shows that $Br_\infty^{(s)}$ lies in the interval $-1 < Br_\infty^{(s)} < 0$ and that, for values of n approximately greater than 4, $Br_\infty^{(s)}$ is almost zero. This result is interesting because, if $Br_\infty = 0$, equation (40) yields a finite and positive value of Nu_∞ for every n. Indeed, if n is roughly greater than 4, this asymptotic value becomes unstable. In fact, due to the singularity, even a very small negative value of Br_∞ can change drastically the value of Nu_∞ and can also produce a change of sign. However, it should be pointed out that the condition $n \geqslant 4$ is of poor physical interest, since it is very hard to find power-law fluids with $n > 3$.

Finally, let us consider a local Brinkman number such that

$$\lim_{X\to+\infty} Br(X) = 0, \quad (42)$$

$$\lim_{X\to+\infty}\frac{1}{Br(X)}\frac{dBr(X)}{dX} = -2\beta, \quad (43)$$

where $\beta \neq 0$ is any positive real number. Equation (42) ensures that the effect of viscous dissipation becomes negligible in the thermally developed region. Equations (42) and (43) are satisfied, for instance, by any axial distribution of wall heat flux which tends to infinity for $x \to +\infty$ and which behaves asymptotically as $e^{2\beta X}$, $Xe^{2\beta X}$, $X^2e^{2\beta X}$, ... etc. In this case, the dimensionless temperature field can be expressed for large values of X by equation (35). By substituting equation (35) in equations (21) and (23) and by employing equations (24), (42) and (43), one obtains in the limit $X \to +\infty$

$$\frac{d}{dR}\left(R\frac{df}{dR}\right) = \frac{3n+1}{2(n+1)}(2^{2-n}+\beta f)\left(1-R^{(n+1)/n}\right)R \quad (44)$$

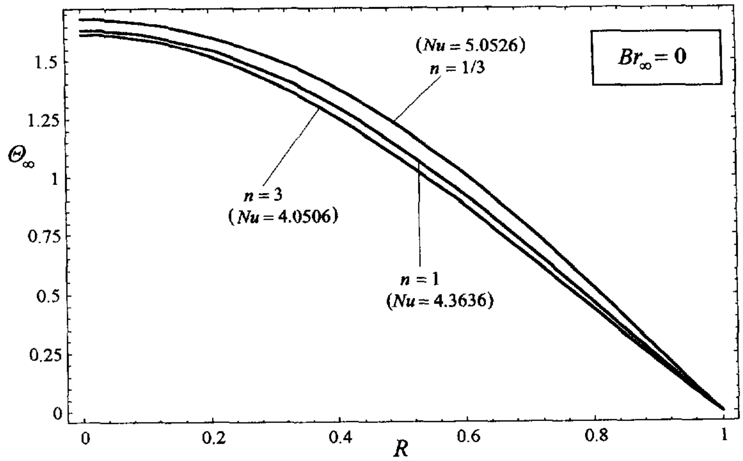

Fig. 3. Θ_∞ vs R for various n and for $Br_\infty = 0$ when the local Brinkman number fulfils equations (33) and (34).

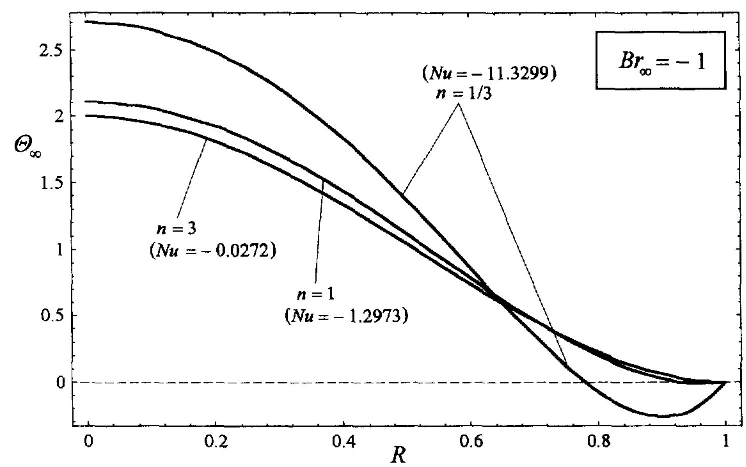

Fig. 4. Θ_∞ vs R for various n and for $Br_\infty = -1$ when the local Brinkman number fulfils equation (33).

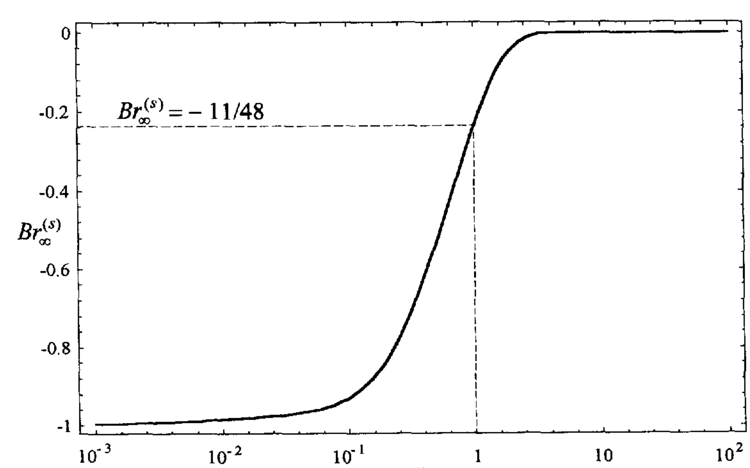

Fig. 5. Plot of $Br_\infty^{(s)}$ vs n.

$$\left.\frac{df}{dR}\right|_{R=0} = 0, \quad \left.\frac{df}{dR}\right|_{R=1} = \frac{1}{2^n}. \quad (45)$$

Equation (44) can be converted into a first-order differential equation by employing the transformation

$$f(R) = \frac{2^{2-n}}{\beta}\left[C\exp\left(\int_0^R a(R')\,dR'\right) - 1\right], \quad (46)$$

where C is a constant which is determined by the boundary condition at $R = 1$. By substituting equation (46) in equation (44), one obtains

$$R\frac{da}{dR} + a + Ra^2 + \frac{3n+1}{n+1}\frac{\beta}{2}\left(R^{(2n+1)/n} - R\right) = 0. \quad (47)$$

Equations (45) and (46) yield

$$a(0) = 0 \quad (48)$$

$$C = \frac{\beta}{4a(1)}\exp\left(-\int_0^1 a(R')\,dR'\right). \quad (49)$$

Since the differential equation (47) is of the first order, equations (47) and (48) uniquely determine the function $a(R)$. Then, equation (49) can be employed to evaluate the constant C, so that equation (46) yields the function $f(R)$. Equation (47) with its boundary condition (48) can be integrated numerically by employing the fourth-order Runge–Kutta method [20]. The function $\Theta_\infty(R)$ can be evaluated by the expression

$$\Theta_\infty(R) = \frac{f(1) - f(R)}{f(1)}, \quad (50)$$

while, on account of equations (9), (45), (46), (49) and (50), the asymptotic value of Nu can be expressed as

$$Nu_\infty = -2\left.\frac{d\Theta_\infty}{dR}\right|_{R=1} = \frac{2\beta a(1)}{\beta - 4a(1)}. \quad (51)$$

The numerical method requires about $2\cdot 10^6$ subdivisions of the interval $0 \leqslant R \leqslant 1$ to obtain, through equation (51), values of Nu_∞ with a precision of six decimal digits. Asymptotic values of Nu evaluated by this method are reported in Table 1. This table shows that, for a fixed value of β, Nu_∞ decreases with n while, for fixed n, Nu_∞ increases with β. In Table 2, the asymptotic values of Nu for $n = 1$ are compared with those obtained by analytical methods in refs. [14, 21, 22] in the case of an exponentially varying wall heat flux. The agreement with the results obtained in refs. [21, 22] is perfect except for $\beta = 10\,000$, where our result coincides with that of Piva [21] and is slightly different from that of Roetzel [22]. On the other hand, the agreement with the results reported in ref. [14] is not as good, due to the low accuracy which can be attained with the method employed by these authors. Indeed, the method employed in ref. [14] to obtain the asymptotic values of Nu is based on the evaluation of a very slowly converging series. Shah and London [14] point out that the sum of the first 10^5 terms of the series is required to get two decimal points accuracy.

Table 1. Asymptotic values of Nu in the case equations (42) and (43) are fulfilled

β	$n = 1/5$	$n = 1/3$	$n = 3$
1	5.6141	5.1431	4.1324
5	5.9774	5.4818	4.4358
10	6.3858	5.8613	4.7715
20	7.0897	6.5129	5.3395
30	7.6865	7.0634	5.8132
40	8.2082	7.5436	6.2230
50	8.6741	7.9717	6.5863
60	9.0969	8.3597	6.9144
70	9.4852	8.7159	7.2146
80	9.8453	9.0459	7.4921
90	10.1818	9.3542	7.7509
100	10.4983	9.6439	7.9938
200	12.9732	11.9067	9.8824
500	17.4740	16.0133	13.2902
1000	22.0461	20.1794	16.7354
10 000	48.3125	44.0751	36.4282

The function $\Theta_\infty(R)$ obtained through equation (50) has been plotted in Figs. 6 and 7 for $n = 1/3$ and $n = 3$, respectively, and for $\beta = 10$, $\beta = 100$, $\beta = 1000$. The behaviour of $\Theta_\infty(R)$ for $n = 1/3$ is almost similar to that for $n = 3$. Moreover, it has been verified that for a Newtonian fluid, i.e. with $n = 1$, the plots of $\Theta_\infty(R)$ can hardly be distinguished from those for $n = 3$. Figures 6 and 7 show that, for fixed n, $\Theta_\infty(R)$ tends to become uniform for increasing values of β. In Figs. 6 and 7, the increase of the slope of $\Theta_\infty(R)$ at $R = 1$ for increasing values of β is clearly represented. Indeed, as a consequence of equation (51), the derivative of $\Theta_\infty(R)$ at $R = 1$ is proportional to Nu_∞ and, as is shown in Tables 1 and 2, Nu_∞ is an increasing function of β.

UNIFORM WALL TEMPERATURE AND CONVECTIVE BOUNDARY CONDITIONS

In this section, the fully developed behaviour of laminar forced convection will be determined, both in the case of uniform wall temperature and in the case of convection, through the duct wall, with an external fluid with a uniform reference temperature T_f and a uniform convection coefficient h_e.

Let us suppose that T_w = constant. Then, on account of equation (20), the wall heat flux can be expressed as

$$q_w = \frac{kNu}{2r_0}(T_w - T_b) = \frac{\eta \bar{u}^{n+1} Nu}{2r_0^n}(\vartheta_w - \vartheta_b). \quad (52)$$

As proved in the Appendix, this wall heat flux fulfils

Table 2. Asymptotic values of Nu for $n = 1$ in the case equations (42) and (43) are fulfilled. The results obtained in this paper are compared with those obtained for exponential wall heat flux in the previous literature

β	Present paper	Shah and London [14]	Roetzel [21]	Piva [22]
1	4.4475	—	4.4475	4.4475
5	4.7596	4.77	—	4.7596
10	5.1067	5.11	5.1067	5.1067
20	5.6974	5.71	—	5.6974
30	6.1925	6.21	—	—
40	6.6222	6.64	—	—
50	7.0040	7.02	—	7.0040
60	7.3493	—	—	—
70	7.6655	—	—	—
80	7.9582	—	—	—
90	8.2313	—	—	—
100	8.4877	—	8.4877	8.4877
200	10.4846	—	—	—
500	14.0951	—	—	—
1000	17.7495	—	17.7495	17.7495
10 000	38.6607	—	38.6674	38.6607

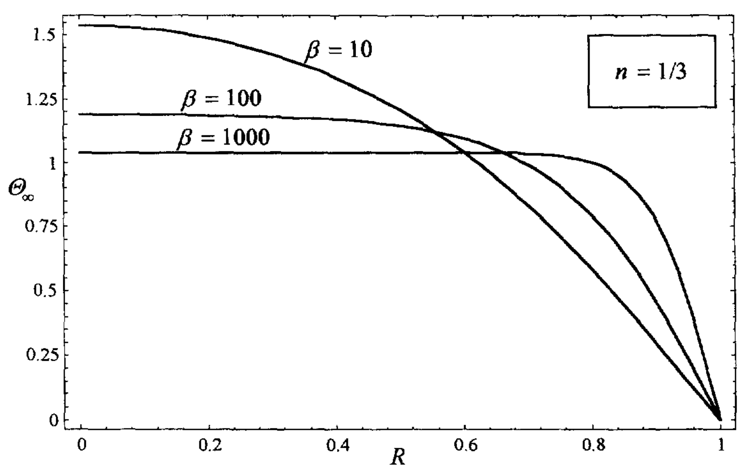

Fig. 6. Θ_∞ vs R for various β and for $n = 1/3$ when the local Brinkman number fulfils equations (42) and (43).

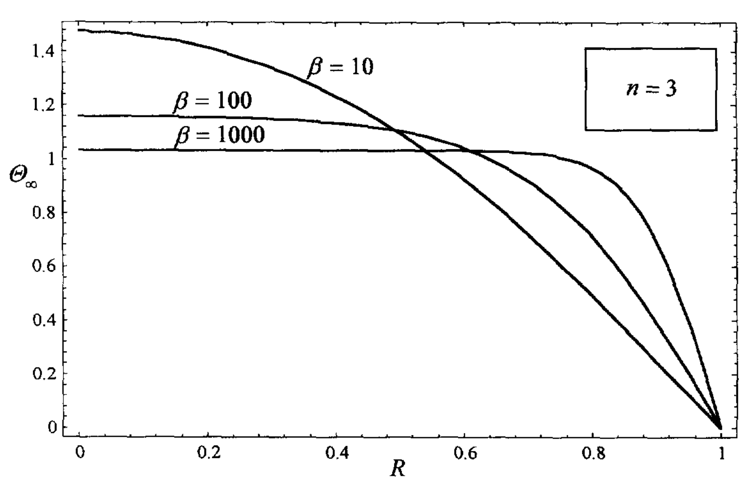

Fig. 7. Θ_∞ vs R for various β and for $n = 3$ when the local Brinkman number fulfils equations (42) and (43).

equation (33) with the nonvanishing value of Br_∞ given by

$$Br_\infty = -2^{-n}\left(\frac{n}{3n+1}\right)^n. \quad (53)$$

Therefore, $\Theta_\infty(R)$ and the asymptotic value of Nu are given by equations (39) and (40), respectively, with Br_∞ expressed by equation (53), i.e.

$$\Theta_\infty(R) = \frac{5n+1}{4n+1}(1-R^{(3n+1)/n}), \quad (54)$$

$$Nu_{T\infty} = \frac{2(3n+1)(5n+1)}{n(4n+1)}. \quad (55)$$

Equation (55) agrees with the results obtained by Dang [11] and in the particular case of a Newtonian fluid yields $Nu_{T\infty} = 48/5$. Both $\Theta_\infty(R)$ and the asymptotic value of Nu depend only on n and are independent of the viscosity of the fluid. Therefore, if T_w = constant, the fully developed behaviour of the temperature field cannot be evaluated by a model which discards the viscous dissipation effects.

Rather similar circumstances occur when the duct wall exchanges heat by convection with an external fluid having a uniform reference temperature T_f and a uniform convection coefficient h_e. In this case, by introducing the Biot number $Bi = h_e r_0/k$, the wall heat flux q_w can be expressed as

$$q_w = \frac{kBi}{r_0}(T_f - T_w) = \frac{\eta \bar{u}^{n+1} Bi}{r_0^n}(\vartheta_f - \vartheta_w). \quad (56)$$

In the Appendix, it is proved that this wall heat flux fulfils equations (33) with the nonvanishing value of Br_∞ expressed by equation (53), as in the case T_w = constant. Therefore, $\Theta_\infty(R)$ and the asymptotic value of Nu are still expressed by equations (54) and (55), respectively. It should be emphasized that $\Theta_\infty(R)$ and Nu_∞ do not depend on the value of Bi and coincide with those obtained in the case T_w = constant. As is proved in the Appendix, the latter can be interpreted as a convective boundary condition with $Bi \to \infty$. Plots of $\Theta_\infty(R)$ evaluated by equation (54) are reported in Fig. 8, for $n = 1/3$, $n = 1$ and $n = 3$. In the same figure, the corresponding values of Nu_∞, evaluated by equation (55), are given. On account of equation (55), it is easily verified that Nu_∞ is a decreasing function of n. Moreover, as a consequence of equation (9), Nu_∞ is proportional to the derivative of $\Theta_\infty(R)$ at $R = 1$. Indeed, Fig. 8 shows that the slope of $\Theta_\infty(R)$ at $R = 1$ decreases when n increases.

CONCLUSIONS

Laminar and hydrodynamically developed forced convection of a power-law fluid flowing in a circular tube with a prescribed axial distribution of wall heat flux has been studied. The effect of viscous dissipation has been taken into account, while the axial heat conduction in the fluid has been considered as negligible. It has been shown that, to ensure the existence of a thermally developed region, it is too restrictive to require that the wall heat flux varies exponentially in the axial direction. Less restrictive conditions are formulated which, if fulfilled by the wall heat flux distribution, are sufficient to yield an asymptotic thermally developed region. This region has been studied in three cases. The first case examined is that of a wall heat flux distribution which tends to zero when $x \to +\infty$. In this case, the asymptotic value of the Nusselt number has been shown to be zero. In the second case, it has been supposed that, when $x \to +\infty$, the wall heat flux distribution $q_w(x)$ does not tend to zero, while $(1/q_w(x))\mathrm{d}q_w(x)/\mathrm{d}x$ tends to zero. Under these assumptions, the asymptotic value of the Nusselt number has been expressed as a function of the power-law index n and of the asymptotic value of the Brinkman number Br_∞. It has been shown that this expression becomes singular when Br_∞ equals a particular function of n. For dilatant fluids with n roughly greater than 4, this function yields values very close

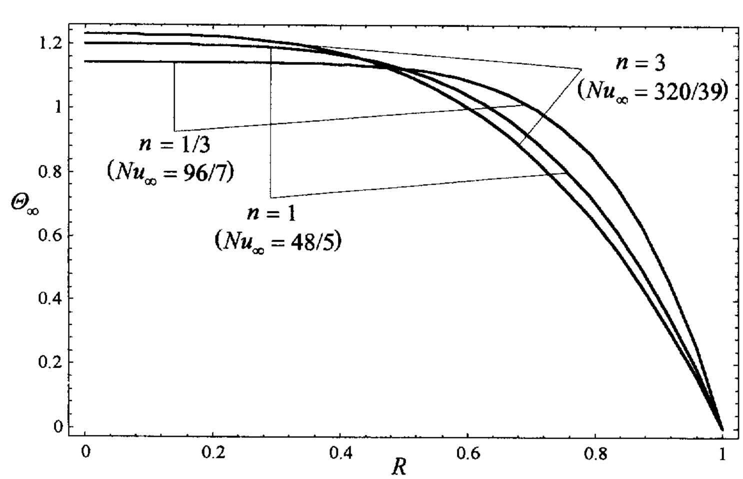

Fig. 8. Θ_∞ vs R for various n in the case of uniform wall temperature and of convective boundary conditions

to zero. Therefore, for $n > 4$ and for $Br_\infty = 0$, the asymptotic behaviour of the temperature field can be considered as unstable. The third case examined in this paper is that of an axial distribution of wall heat flux such that, when $x \to +\infty$, $q_w(x)$ tends to infinity, while $(1/q_w(x))\, dq_w(x)/dx$ tends to a positive constant. If these conditions are fulfilled, the effect of viscous dissipation becomes negligible in the thermally developed region and the asymptotic value of the Nusselt number is a function of n and of a dimensionless parameter β. The asymptotic behaviour of the function $\Theta = (T_w - T)/(T_w - T_b)$ and the asymptotic values of Nu have been evaluated numerically for some values of n and of β.

Finally, it has been shown that both the boundary condition of uniform wall temperature and that of convection with an external isothermal fluid can be reduced, in the thermally developed region, to the second of the three cases quoted above. In particular, it has been proved that the fully developed value of Nu for convective boundary conditions is independent of the Biot number and is equal to the value of Nu_∞ for uniform wall temperature.

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APPENDIX

In the case of uniform wall temperature, equations (22), (24) and (52) yield

$$\frac{d}{dX}(\vartheta_w - \vartheta_b) = -4Nu(\vartheta_w - \vartheta_b) - 8\left(\frac{3n+1}{n}\right)^n. \quad (A1)$$

If the temperature field is fully developed, $Nu = Nu_\infty$ and equation (A1) can be easily integrated. Therefore, one obtains

$$\vartheta_w - \vartheta_b(X) = \vartheta_w \exp(-4Nu_\infty X) - \frac{2}{Nu_\infty}\left(\frac{3n+1}{n}\right)^n. \quad (A2)$$

By substituting equation (A2) in equation (52), in the fully developed region q_w can be expressed as

$$q_w(X) = \frac{\eta \bar{u}^{n+1} Nu_\infty}{2r_0^n}\vartheta_w \exp(-4Nu_\infty X) - \frac{\eta \bar{u}^{n+1}}{r_0^n}\left(\frac{3n+1}{n}\right)^n. \quad (A3)$$

On account of equation (A3), it is easily verified that condition (33) is satisfied with the nonvanishing value of Br_∞ given by

$$Br_\infty = -2^{-n}\left(\frac{n}{3n+1}\right)^n. \quad (A4)$$

In the case of external convection with an isothermal fluid, on account of equations (22), (52) and (56), the local Brinkman number is given by

$$Br(X) = \frac{Nu+2Bi}{2^n NuBi(\vartheta_f-\vartheta_b)}. \quad (A5)$$

By substituting equation (A5) in equation (24), one obtains

$$\frac{d}{dX}(\vartheta_f-\vartheta_b) = -\frac{8NuBi}{Nu+2Bi}(\vartheta_f-\vartheta_b)-8\left(\frac{3n+1}{n}\right)^n. \quad (A6)$$

In the fully developed region $Nu = Nu_\infty$ so that equation (A6) can be easily integrated and yields

$$\vartheta_f-\vartheta_b(X) = \vartheta_f \exp\left(-\frac{8Nu_\infty Bi}{Nu_\infty+2Bi}X\right) - \frac{Nu_\infty+2Bi}{Nu_\infty Bi}\left(\frac{3n+1}{n}\right)^n. \quad (A7)$$

As a consequence of equations (A5) and (A7), one obtains, for large values of X,

$$Br(X) = 2^{-n}\left\{\frac{Nu_\infty Bi}{Nu_\infty+2Bi}\vartheta_f \times \exp\left(-\frac{8Nu_\infty Bi}{Nu_\infty+2Bi}X\right)-\left(\frac{3n+1}{n}\right)^n\right\}^{-1}. \quad (A8)$$

It is easily verified that $Br(X)$ given by equation (A8) fulfils equation (33) with Br_∞ expressed by equation (A4) as in the case T_w = constant. The latter can be interpreted as a convective boundary condition with $Bi \to \infty$. In fact, equation (A8) ensures that, in the limit $Bi \to \infty$, the wall heat flux has a finite value. Therefore, on account of equation (56), T_w must tend to T_f when $Bi \to \infty$.